

## ON THE SIMILARITY OF HOLOMORPHIC MATRICES

JÜRGEN LEITERER

*Dedicated to the memory of Gennadi Henkin, teacher and coauthor*

ABSTRACT. R. Guralnick [Gu] proved that two holomorphic matrices on a noncompact connected Riemann surface, which are locally holomorphically similar, are globally holomorphically similar. We generalize this to (possibly, non-smooth) one-dimensional Stein spaces. For Stein spaces of arbitrary dimension, we prove that global  $C^\infty$  similarity implies global holomorphic similarity, whereas global continuous similarity is not sufficient.

## 1. INTRODUCTION

Let  $X$  be a complex space, e.g., a complex manifold or an analytic subset of a complex manifold. Denote by  $\text{Mat}(n \times n, \mathbb{C})$  the algebra of complex  $n \times n$  matrices, and by  $\text{GL}(n, \mathbb{C})$  the group of its invertible elements.

**1.1. Definition.** Two holomorphic maps  $A, B : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$  are called (globally) **holomorphically similar on  $X$**  if there is a holomorphic map  $H : X \rightarrow \text{GL}(n, \mathbb{C})$  with  $B = H^{-1}AH$  on  $X$ . They are called **locally holomorphically similar at a point  $\xi \in X$**  if there is a neighborhood  $U$  of  $\xi$  such that  $A|_U$  and  $B|_U$  are holomorphically similar on  $U$ . Correspondingly, **continuous** and  **$C^k$  similarity** are defined.

R. Guralnick [Gu] proved the following

**1.2. Theorem.** *Suppose  $X$  is a noncompact connected Riemann surface, and let  $A, B : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$  be two holomorphic maps, which are locally holomorphically similar at each point of  $X$ . Then  $A$  and  $B$  are globally holomorphically similar on  $X$ .*

Actually, Guralnick proved a more general theorem for matrices with elements in a Bezout ring (with some extra properties), and then applies this to the ring of holomorphic functions on a non-compact connected Riemann surface. The ring of holomorphic functions on an arbitrary (non-smooth) one-dimensional Stein space is not Bezout. Therefore, it seems that Guralnick's proof of Theorem 1.2 cannot be generalized to the non-smooth case, at least not in a straightforward way.

Nevertheless, in the present paper, we use Guralnick's result to prove

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**1.3. Theorem.** *Suppose  $X$  is a one-dimensional Stein space (possibly not smooth), and let  $A, B : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$  be two holomorphic maps, which are locally holomorphically similar at each point of  $X$ . Then  $A$  and  $B$  are globally holomorphically similar on  $X$ .*

In the proof (given in Section 5) we take advantage of the fact that the normalization of  $X$  is a Riemann surface each connected component of which is noncompact, and there, we can apply Guralnick's result. Then we use the Oka principle for Oka pairs of Forster and Ramspott [FR1] (Proposition 4.6 below) to “push down” this to  $X$ .

The Oka principle of Forster and Ramspott is valid also for Stein spaces of arbitrary dimension. We deduce from it (in Section 4) the following Oka principle for the similarity of holomorphic matrices.

**1.4. Theorem.** *Suppose  $X$  is a Stein space (of arbitrary dimension), and let  $A, B : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$  be two holomorphic maps such that there exists a continuous map  $C : X \rightarrow \text{GL}(n, \mathbb{C})$  satisfying the following two conditions:*

- (a)  $B = C^{-1}AC$  on  $X$ .
- (b) *For each  $\xi \in X$ , there exist a neighborhood  $U_\xi$  of  $\xi$  and a holomorphic map  $H_\xi : U_\xi \rightarrow \text{GL}(n, \mathbb{C})$  with  $B = H_\xi^{-1}AH_\xi$  on  $U_\xi$  and  $H_\xi(\xi) = C(\xi)$ .*

*Then  $A$  and  $B$  are globally holomorphically similar on  $X$ .*

Note that conditions (a) and (b) imply that

- (a')  $A$  and  $B$  are globally continuously similar on  $X$ .
- (b')  $A$  and  $B$  are locally holomorphically similar at each point of  $X$ .

However, conditions (a') and (b') alone do not imply global holomorphic similarity. We show this by a counterexample (Theorem 8.2 below).

There are different criteria for local holomorphic similarity, which are known or can be easily obtained from known results. They are contained in the following theorem (with invertible  $\Phi$ ).

**1.5. Theorem.** *Let  $A, B : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$  be holomorphic,  $\xi \in X$  and  $\Phi \in \text{Mat}(n \times n, \mathbb{C})$  such that  $\Phi B(\xi) = A(\xi)\Phi$ . Suppose at least one of the following conditions is satisfied.*

- (i) (Wasow's criterion) *The dimension of the complex vector space*

$$(1.1) \quad \left\{ \Theta \in \text{Mat}(n \times n, \mathbb{C}) \mid \Theta B(\zeta) = A(\zeta)\Theta \right\}$$

*is constant for  $\zeta$  in some neighborhood of  $\xi$ .*

- (ii) (Smith's criterion)  $\dim X = 1$ ,  $\xi$  is a smooth point of  $X$ , and there exist a neighborhood  $V_\xi$  of  $\xi$  and a continuous map  $C_\xi : V_\xi \rightarrow \text{GL}(n, \mathbb{C})$  such that  $C_\xi B = AC_\xi$  on  $V_\xi$ , and  $C_\xi(\xi) = \Phi$ .
- (iii) (Spallek's criterion) *There exist a neighborhood  $V_\xi$  of  $\xi$  and a  $C^\infty$  map  $T_\xi : V_\xi \rightarrow \text{Mat}(n \times n, \mathbb{C})$  such that  $T_\xi B = AT_\xi$  on  $V_\xi$ , and  $T_\xi(\xi) = \Phi$ .*

*Then there exist a neighborhood  $U_\xi$  of  $\xi$  and a holomorphic  $H_\xi : U_\xi \rightarrow \text{Mat}(n \times n, \mathbb{C})$  such that  $H_\xi B = AH_\xi$  on  $U_\xi$ , and  $H_\xi(\xi) = \Phi$ .*

Proofs or references (explaining also the role of the names ‘Wasow, Smith and Spallek’) for the statements contained in this theorem will be given in Section 6.

From Spallek's criterion it follows that each  $C^\infty$  map  $C : X \rightarrow \text{GL}(n, \mathbb{C})$  satisfying condition (a) in Theorem 1.4 automatically also satisfies condition (b). Therefore, Theorem 1.4 has the following

**1.6. Corollary.** *Suppose  $X$  is a Stein space (of arbitrary dimension). Let  $A, B : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$  be two holomorphic maps, which are globally  $\mathcal{C}^\infty$  similar on  $X$ . Then  $A$  and  $B$  are globally holomorphically similar on  $X$ .*

We show by an example (Theorem 8.2 below) that in this corollary  $\mathcal{C}^\infty$  cannot be replaced by  $\mathbb{C}^k$  with  $k < \infty$  (the same  $k$  for all  $A, B$ ).

Moreover, Spallek's criterion in particular says that local  $\mathcal{C}^\infty$  similarity at a point implies local holomorphic similarity at this point, and the Smith criterion says that, if  $\dim X = 1$ , then, at the smooth points, merely local continuous similarity implies local holomorphic similarity. Therefore Theorem 1.3 has the following

**1.7. Corollary.** *Suppose  $X$  is a one-dimensional Stein space. Let  $A, B : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$  be holomorphic. Then, for global holomorphic similarity of  $A$  and  $B$  it is sufficient that, for each point  $\xi \in X$ , at least one of the following holds.*

- $A$  and  $B$  are locally  $\mathcal{C}^\infty$  similar at  $\xi$ .
- $\xi$  is a smooth point of  $X$ , and  $A$  and  $B$  are locally continuously similar at  $\xi$ .

We show by examples (Theorem 7.4) that in this corollary, at the non-smooth points,  $\mathcal{C}^\infty$  cannot be replaced by  $\mathcal{C}^k$  with  $k < \infty$  (the same  $k$  for all  $A, B$  and  $\xi$ ). However, see Remark 6.4.

In a forthcoming paper we will give another proof of Theorem 1.3, which does not use Guralnick's Theorem 1.2 (and thereby is also a new proof for Guralnick's result), but which is much longer than the proof given here. An advantage of this other proof is that it is not restricted to the one-dimensional case.

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## 2. NOTATIONS

$\mathbb{N}$  is the set of natural numbers including 0.  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ .

If  $n, m \in \mathbb{N}^*$ , then by  $\text{Mat}(n \times m, \mathbb{C})$  we denote the space of complex  $n \times m$  matrices ( $n$  rows and  $m$  columns), and by  $\text{GL}(n, \mathbb{C})$  we denote the group of invertible complex  $n \times n$  matrices.

The unit matrix in  $\text{Mat}(n \times n, \mathbb{C})$  will be denoted by  $I_n$  or simply by  $I$ .

$\text{Ker } \Phi$  denotes the kernel,  $\text{Im } \Phi$  the image and  $\|\Phi\|$  the operator norm of a matrix  $\Phi \in \text{Mat}(n \times m, \mathbb{C})$  considered as a linear map between the Euclidean spaces  $\mathbb{C}^m$  and  $\mathbb{C}^n$ .

By a complex space we always mean a *reduced* complex space [GR] (which is the same as an *analytic space* in the terminology used, e.g., in [C] and [L]).

## 3. PREPARATIONS CONCERNING SHEAVES

Let  $X$  be a topological space, and  $G$  a topological group (abelian or non-abelian). Then we denote by  $\mathcal{C}_X^G$ , or simply by  $\mathcal{C}^G$ , the sheaf of continuous  $G$ -valued maps on  $X$ , i.e.,  $\mathcal{C}^G$  is the map which assigns to each open  $U \subseteq X$  the group  $\mathcal{C}^G(U)$  of all continuous maps  $f : U \rightarrow G$  if  $U \neq \emptyset$ , and the group which consist only of the neutral element of  $G$  if  $U = \emptyset$ .

All sheaves in this paper are subsheaves of  $\mathcal{C}_X^G$  (for some  $X$  and some  $G$ ), i.e., a map  $\mathcal{F}$  which assigns to each open  $U \subseteq X$  a subgroup  $\mathcal{F}(U)$  of  $\mathcal{C}^G(U)$  such that:

- If  $V \subseteq U$  are non-empty open subsets of  $X$ , then, for each  $f \in \mathcal{F}(U)$ , the restriction of  $f$  to  $V$ ,  $f|_V$ , belongs to  $\mathcal{F}(V)$ .

- If  $U \subseteq X$  is open and  $f \in \mathcal{C}^G(U)$  is such that, for each  $\xi \in U$ , there is an open neighborhood  $V \subseteq U$  of  $\xi$  with  $f|_V \in \mathcal{F}(V)$ , then  $f \in \mathcal{F}(U)$ .

If  $\mathcal{F}$  and  $\mathcal{G}$  are two subsheaves of  $\mathcal{C}_X^G$ , then  $\mathcal{F}$  is called a subsheaf of  $\mathcal{G}$  if  $\mathcal{F}(U) \subseteq \mathcal{G}(U)$  for all open  $U \subseteq X$ .

If  $X$  is a complex space and  $G$  is a complex Lie group, then we denote by  $\mathcal{O}_X^G$ , or simply by  $\mathcal{O}^G$ , the subsheaf of  $\mathcal{C}_X^G$  which assigns to each non-empty open  $U \subseteq X$ , the group  $\mathcal{O}^G(U)$  of all holomorphic maps from  $U$  to  $G$ .

Let  $\mathcal{F}$  be a subsheaf of  $\mathcal{C}^G$ .

Let  $\mathcal{U} = \{U_i\}_{i \in I}$  an open covering of  $X$ .

A family  $f_{ij} \in \mathcal{F}(U_i \cap U_j)$ ,  $i, j \in I$ , is called a  $(\mathcal{U}, \mathcal{F})$ -**cocycle** if (with the group operation in  $G$  written as a multiplication)

$$f_{ij}f_{jk} = f_{ik} \quad \text{on } U_i \cap U_j \cap U_k \quad \text{for all } i, j, k \in I. \quad {}^1$$

Note that then always  $f_{ij}^{-1} = f_{ji}$  and  $f_{ii}$  is identically equal to the neutral element of  $G$ . The set of all  $(\mathcal{U}, \mathcal{F})$ -cocycles will be denoted by  $Z^1(\mathcal{U}, \mathcal{F})$ . Two cocycles  $\{f_{ij}\}$  and  $\{g_{ij}\}$  in  $Z^1(\mathcal{U}, \mathcal{F})$  are called  **$\mathcal{F}$ -equivalent** if there exists a family  $h_i \in \mathcal{F}(U_i)$ ,  $i \in I$ , such that

$$f_{ij} = h_i g_{ij} h_j^{-1} \quad \text{on } U_i \cap U_j \quad \text{for all } i, j \in I.$$

If, in this case, for all  $i, j$ , the map  $g_{ij}$  is identically equal to the neutral element of  $G$ , then  $f$  is called  **$\mathcal{F}$ -trivial**.

We say that  $f$  is an  **$\mathcal{F}$ -cocycle** (on  $X$ ), if there exists an open covering  $\mathcal{U}$  of  $X$  with  $f \in Z^1(\mathcal{U}, \mathcal{F})$ . This covering then is called **the covering of  $f$** . As usual we write

$$H^1(X, \mathcal{F}) = 0$$

to say that each  $\mathcal{F}$ -cocycle is  $\mathcal{F}$ -trivial.

Let  $\mathcal{U} = \{U_i\}_{i \in I}$  and  $\mathcal{U}^* = \{U_\alpha^*\}_{\alpha \in I^*}$  be two open coverings of  $X$  such that  $\mathcal{U}^*$  is a refinement of  $\mathcal{U}$ , i.e., there is a map  $\tau : I^* \rightarrow I$  with  $U_\alpha^* \subseteq U_{\tau(\alpha)}$  for all  $\alpha \in I^*$ . Then we say that a  $(\mathcal{U}^*, \mathcal{F})$ -cocycle  $\{f_\alpha^*\}_{\alpha \in I^*}$  is **induced** by a  $(\mathcal{U}, \mathcal{F})$ -cocycle  $\{f_{ij}\}_{i, j \in I}$  if this map  $\tau$  can be chosen so that

$$f_{\alpha\beta}^* = f_{\tau(\alpha)\tau(\beta)} \quad \text{on } U_i^* \cap U_j^* \quad \text{for all } \alpha, \beta \in I^*.$$

We need the following well-known proposition, see [C, p. 101] for “if” and [Hi, p. 41] for “only if”.

**3.1. Proposition.** *Let  $f, g \in Z^1(\mathcal{U}, \mathcal{F})$  and  $f^*, g^* \in Z^1(\mathcal{U}^*, \mathcal{F})$  such that  $f^*$  and  $g^*$  are induced by  $f$  and  $g$ , respectively. Then  $f$  and  $g$  are  $\mathcal{F}$ -equivalent if and only if  $f^*$  and  $g^*$  are  $\mathcal{F}$ -equivalent.*

Now let  $\mathcal{U}$  and  $\mathcal{V}$  be two arbitrary open coverings of  $X$ ,  $f \in Z^1(\mathcal{U}, \mathcal{F})$  and  $g \in Z^1(\mathcal{V}, \mathcal{F})$ . Then we say that  $f$  and  $g$  are  **$\mathcal{F}$ -equivalent** if there exist an open covering  $\mathcal{W}$  of  $X$ , which is a refinement of both  $\mathcal{U}$  and  $\mathcal{V}$ , and  $(\mathcal{W}, \mathcal{F})$  cocycles  $f^*$  and  $g^*$ , which are induced by  $f$  and  $g$ , respectively, such that  $f^*$  and  $g^*$  are  $\mathcal{F}$ -equivalent. By Proposition 3.1, this definition is in accordance with the definition of equivalence given above for  $\mathcal{U} = \mathcal{V}$ .

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<sup>1</sup>Here and in the following we use the convention that statements like “ $f = g$  on  $\emptyset$ ” or “ $f := g$  on  $\emptyset$ ” have to be omitted.

## 4. AN OKA PRINCIPLE AND PROOF OF THEOREM 1.4

**4.1. Definition.** Let  $\Phi \in \text{Mat}(n \times n, \mathbb{C})$ . We denote by  $\text{Com } \Phi$  the algebra of all  $\Theta \in \text{Mat}(n \times n, \mathbb{C})$  with  $\Phi\Theta = \Theta\Phi$ , and by  $\text{GCom } \Phi$  we denote the group of invertible elements of  $\text{Com } \Phi$ . Note that, as easily seen,

$$(4.1) \quad \text{GCom } \Phi = \text{GL}(n, \mathbb{C}) \cap \text{Com } \Phi, \text{ and}$$

$$(4.2) \quad \text{Com } (\Gamma^{-1}\Phi\Gamma) = \Gamma^{-1}(\text{Com } \Phi)\Gamma \quad \text{for all } \Gamma \in \text{GL}(n, \mathbb{C}).$$

**4.2. Lemma.**  $\text{GCom } \Phi$  is connected, for each  $\Phi \in \text{Mat}(n \times n, \mathbb{C})$ .

*Proof.* Let  $\Theta \in \text{GCom } \Phi$ . Since the set of eigenvalues of  $\Phi$  is finite, and the numbers 0 and  $-1 - \|\Theta\|$  do not belong to it, then we can find a continuous map  $\lambda : [0, 1] \rightarrow \mathbb{C}$  such that  $\lambda(0) = 0$ ,  $\lambda(1) = 1 + \|\Theta\|$  and  $\Theta + \lambda(t)I \in \text{GL}(n, \mathbb{C})$  for all  $0 \leq t \leq 1$ . Setting

$$\gamma(t) = \begin{cases} \Theta + \lambda(t)I & \text{if } 0 \leq t \leq 1, \\ (1 + \|\Theta\|) \left( \frac{2-t}{1+\|\Theta\|} \Theta + I \right) & \text{if } 1 \leq t \leq 2, \\ (1 + (3-t)\|\Theta\|)I & \text{if } 2 \leq t \leq 3, \end{cases}$$

then we obtain a continuous path  $\gamma$  in  $\text{GL}(n, \mathbb{C})$ , which connects  $\Theta = \gamma(0)$  with  $I = \gamma(3)$ . Since  $\Theta \in \text{Com } \Phi$ , from the definition of  $\gamma$  it is clear that the values of  $\gamma$  belong to the algebra  $\text{Com } \Phi$ . In view of (4.1), this means that  $\gamma$  lies inside  $\text{GCom } \Phi$ .  $\square$

**4.3. Definition.** Let  $X$  be a complex space, and  $A : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$  holomorphic. We introduce the families

$$\text{Com } A := \{\text{Com } A(z)\}_{z \in X} \quad \text{and} \quad \text{GCom } A := \{\text{GCom } A(z)\}_{z \in X}.$$

If the dimension of  $\text{Com } A(z)$  does not depend on  $z$ , then it follows from the Wasow criterion (Theorem 1.5, condition (i)) that  $\text{Com } A$  is a holomorphic vector bundle, but it is clear that this dimension can jump (in an analytic set). But even if  $\text{Com } A$  is a holomorphic vector bundle,  $\text{Com } A$  need not be locally trivial as a bundle of algebras. In particular,  $\text{GCom } A$  need not be locally trivial as a bundle of groups. Moreover, it is possible that  $\text{GCom } A$  is not locally trivial as a bundle of topological spaces. We give an example.

**4.4. Example.** Let  $X = \mathbb{C}$  and  $A(z) := \begin{pmatrix} z & 1 \\ 0 & 0 \end{pmatrix}$ ,  $z \in \mathbb{C}$ . Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Com } A(z) \Leftrightarrow \begin{pmatrix} za + c & zb + d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} za & a \\ zc & c \end{pmatrix} \Leftrightarrow c = 0 \text{ and } a = zb + d,$$

which implies that  $\dim \text{Com } A(z) = 2$  for all  $z \in \mathbb{C}$ . However

$$\text{GCom } A(0) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in \mathbb{C}^*, b \in \mathbb{C} \right\}$$

whereas, for  $z \neq 0$ ,  $\text{GCom } A(z)$  is isomorphic to

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{C}^* \right\} \quad \text{if } z \neq 0,$$

which implies that  $\pi_1(\text{GCom } A(0)) = \mathbb{Z}$  whereas  $\pi_1(\text{GCom } A(z)) = \mathbb{Z}^2$  if  $z \neq 0$ . Hence, for  $z \neq 0$ ,  $\text{GCom } A(z)$  is not homeomorphic to  $\text{GCom } A(0)$ .

**4.5. Definition.** Let  $X$  be a complex space, and  $A : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$  holomorphic. Even if the families  $\text{Com } A$  and/or  $\text{GCom } A$  are not locally trivial, their sheaves of holomorphic and continuous sections are well-defined. We denote them by  $\mathcal{O}^{\text{Com } A}$ ,  $\mathcal{O}^{\text{GCom } A}$ ,  $\mathcal{C}^{\text{Com } A}$  and  $\mathcal{C}^{\text{GCom } A}$ , respectively.

Further, we define subsheaves  $\widehat{\mathcal{C}}^{\text{Com } A}$  and  $\widehat{\mathcal{C}}^{\text{GCom } A}$  of  $\mathcal{C}^{\text{Com } A}$  and  $\mathcal{C}^{\text{GCom } A}$ , respectively, as follows: if  $U$  is a non-empty open subset of  $X$ , then  $\widehat{\mathcal{C}}^{\text{Com } A}(U)$  is the algebra of all continuous maps  $f : U \rightarrow \text{Mat}(n \times n, \mathbb{C})$  such that, for each  $\xi \in U$ , the following condition is satisfied:

$$(4.3) \quad \left\{ \begin{array}{l} \text{there exist a neighborhood } V_\xi \text{ of } \xi \\ \text{and } h_\xi \in \mathcal{O}^{\text{Com } A}(V_\xi) \text{ with } h(\xi) = f(\xi), \end{array} \right.$$

and we set  $\widehat{\mathcal{C}}^{\text{GCom } A}(U) = \mathcal{C}^{\text{GL}(n, \mathbb{C})}(U) \cap \widehat{\mathcal{C}}^{\text{Com } A}(U)$ .

The following proposition is a special case of the Oka principle for Oka pairs of O. Forster and K. J. Ramsrott [FR1, Satz 1].

**4.6. Proposition.** *Let  $X$  be a Stein space, and  $A : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$  holomorphic. Then each  $\widehat{\mathcal{C}}^{\text{GCom } A}$ -trivial  $\mathcal{O}^{\text{GCom } A}$ -cocycle is  $\mathcal{O}^{\text{GCom } A}$ -trivial.*

Indeed, it is easy to see that, for each non-empty open  $U \subseteq X$  we have: If  $h \in \mathcal{O}^{\text{Com } A}(U)$ , then  $e^h \in \mathcal{O}^{\text{GCom } A}(U)$ , and, if  $H \in \mathcal{O}^{\text{GCom } A}(U)$  with  $\sup_{\zeta \in U} \|H(\zeta) - I\| < 1$ , then

$$\log H := \sum_{\mu=1}^{\infty} (-1)^{\mu-1} \frac{(H - I)^\mu}{\mu} \in \mathcal{O}^{\text{Com } A}(U).$$

This shows that  $\mathcal{O}^{\text{GCom } A}$  is a *coherent  $\mathcal{O}$ -subsheaf* of  $\mathcal{O}_X^{\text{GL}(n, \mathbb{C})}$  in the sense of [FR1, §2], where  $\mathcal{O}^{\text{Com } A}$  is the *generating sheaf of Lie algebras*. Moreover, as observed in [FR1, §2.3, example b)], the pair  $(\mathcal{O}^{\text{GCom } A}, \widehat{\mathcal{C}}^{\text{GCom } A})$  is an *admissible pair* in the sense of [FR1], which, trivially, satisfies condition (PH) in Satz 1 of [FR1]). Therefore the proposition follows from that Satz 1.  $\square$

**4.7. Proof of Theorem 1.4:** Since  $A$  and  $B$  are locally holomorphically similar at each point of  $X$ , we can find an open covering  $\{U_i\}_{i \in I}$  of  $X$  and holomorphic maps  $H_i : U_i \rightarrow \text{GL}(n, \mathbb{C})$  such that

$$(4.4) \quad B = H_i^{-1} A H_i \quad \text{on } U_i.$$

Then  $H_i^{-1} A H_i = B = H_j^{-1} A H_j$  on  $U_i \cap U_j$ . Hence  $A H_i H_j^{-1} = H_i H_j^{-1} A$  on  $U_i \cap U_j$ , i.e., the family

$$(4.5) \quad \{H_i H_j^{-1}\}_{i, j \in I}$$

is an  $\mathcal{O}^{\text{GCom } A}$ -cocycle.

Now, by hypothesis, we have a continuous map  $C : X \rightarrow \text{GL}(n, \mathbb{C})$  satisfying conditions (a) and (b) in Theorem 1.4. Set  $c_i = H_i C$  on  $U_i$ . We claim that

$$(4.6) \quad c_i \in \widehat{\mathcal{C}}^{\text{GCom } A}(U_i).$$

Indeed, let  $\xi \in U_i$  be given. Then, by condition (b), we can find a neighborhood  $V_\xi$  of  $\xi$  and a holomorphic map  $H_\xi : V_\xi \rightarrow \text{GL}(n, \mathbb{C})$  with

$$(4.7) \quad B = H_\xi^{-1} A H_\xi \quad \text{on } V_\xi, \quad \text{and}$$

$$(4.8) \quad H_\xi(\xi) = C(\xi).$$

Set  $h = H_i H_\xi^{-1}$  on  $U_i \cap V_\xi$ . Then from (4.7) and (4.4) it follows that

$$hAh^{-1} = H_i H_\xi^{-1} A H_\xi H_i^{-1} = H_i B H_i^{-1} = A \quad \text{on } U_i \cap V_\xi,$$

i.e.,  $h \in \mathcal{O}^{\text{GCom } A}(U_i \cap V_\xi)$ , and from (4.8) we see that

$$h(\xi) = H_i(\xi) C(\xi)^{-1} = c_i(\xi).$$

which proves (4.6).

Moreover

$$c_i c_j^{-1} = H_i C C^{-1} H_j^{-1} = H_i H_j^{-1} \quad \text{on } U_i \cap U_j.$$

Together with (4.6) this shows that the cocycle (4.5) is  $\widehat{\mathcal{C}}^{\text{GCom } A}$ -trivial. By Proposition 4.6) this means that this cocycle is even is  $\mathcal{O}^{\text{GCom } A}$ -trivial, i.e.,  $H_i H_j^{-1} = h_i h_j^{-1}$  on  $U_i \cap U_j$ , for some family  $h_i \in \mathcal{O}^{\text{GCom } A}(U_i)$ . Then  $h_i^{-1} H_i = h_j^{-1} H_j$  on  $U_i \cap U_j$ . Hence, there is a well-defined global holomorphic map  $H : X \rightarrow \text{GL}(n, \mathbb{C})$  with  $H = H_i^{-1} h_i$  on  $U_i$ , and which satisfies, by (4.4),  $H^{-1} B H = h_i^{-1} H_i B H_i^{-1} h_i = h_i^{-1} A h_i = A$  on  $X$ .  $\square$

## 5. PROOF OF THEOREM 1.3

**5.1. Lemma.** *Let  $X$  be a complex space,  $A : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$  a holomorphic map,  $\Lambda \subseteq X$  a finite set,  $U$  a neighborhood of  $\Lambda$  and  $f \in \widehat{\mathcal{C}}^{\text{GCom } A}(U)$  (Def. 4.5). Then there exists a neighborhood  $W_1 \subseteq U$  of  $\Lambda$  such that, for each neighborhood  $W_2$  of  $\Lambda$  with  $\overline{W}_2 \subseteq W_1$ , there is a map  $\tilde{f} \in \widehat{\mathcal{C}}^{\text{GCom}(A)}(X)$  with  $\tilde{f} = f$  on  $W_2$  and  $\tilde{f} = I$  on  $X \setminus W_1$ .*

*Proof.* We may assume that  $\Lambda$  consist only of one point,  $\xi$ . Choose a neighborhood  $V$  of  $\xi$  so small that  $\overline{V}$  is compact and contained in  $U$ , and set

$$\alpha = 1 + \max_{\zeta \in \overline{V}} \|f(\zeta)\|.$$

Then  $-\alpha$  is not an eigenvalue of  $f(\xi)$ . Moreover, 0 is not an eigenvalue of  $f(\xi)$  (as  $f(\xi)$  is invertible). Therefore, we can find a continuous function  $\lambda : [-1, 0] \rightarrow \mathbb{C}$  such that  $\lambda(-1) = 0$ ,  $\lambda(0) = \alpha$  and  $f(\xi) + \lambda(t)I \in \text{GL}(n, \mathbb{C})$  for all  $-1 \leq t \leq 0$ . Since  $f$  is continuous, we can choose a neighborhood  $W_1 \subseteq V$  of  $\xi$  so small that  $\overline{W}_1 \subseteq V$  and

$$f(\zeta) + \lambda(t)I \in \text{GL}(n, \mathbb{C}) \quad \text{for all } -1 \leq t \leq 0 \text{ and } \zeta \in \overline{W}_1.$$

Then, setting, for  $\zeta \in \overline{W}_1$ ,

$$g(t, \zeta) = \begin{cases} f(\zeta) + \lambda(t)I & \text{if } -1 \leq t \leq 0, \\ \left(1 - t + \frac{t}{\alpha}\right) \left(f(\zeta) + \alpha I\right) & \text{if } 0 \leq t \leq 1, \end{cases}$$

we obtain a continuous map  $g : [-1, 1] \times \overline{W}_1 \rightarrow \text{GL}(n, \mathbb{C})$  (recall that, by definition,  $\alpha > 1$  and therefore  $1 - t + \frac{t}{\alpha} \neq 0$  for all  $t \geq 1$ ) such that

$$\begin{aligned} g(-1, \zeta) &= f(\zeta) \quad \text{and} \quad g(1, \zeta) = \frac{1}{\alpha} f(\zeta) + I \quad \text{for all } \zeta \in \overline{W}_1, \text{ and,} \\ g(t, \cdot) &\in \widehat{\mathcal{C}}^{\text{GCom}(A)}(V) \quad \text{for all } -1 \leq t \leq 1. \end{aligned}$$

Moreover, it follows from the definition of  $\alpha$  that

$$\|g(1, \zeta) - I\| < 1 \quad \text{for all } \zeta \in \overline{W}_1.$$

Choose  $-1 = t_1 < t_2 < \dots < t_m = 1$  such that

$$\|g(t_j, \zeta)g(t_{j+1}, \zeta)^{-1} - I\| < 1 \quad \text{for all } \zeta \in \overline{W_1} \text{ and } 1 \leq j \leq m-1,$$

and define, for  $\zeta \in V$ ,

$$g_m(\zeta) = g(1, \zeta) \quad \text{and} \quad g_j(\zeta) = g(t_j, \zeta)g(t_{j+1}, \zeta)^{-1} \quad \text{if } 1 \leq j \leq m-1.$$

Then  $g_j \in \widehat{\mathcal{C}}^{\text{GCom}(A)}(W_1)$  and  $\|g_j - I\| < 1$  on  $W_1$ , for  $1 \leq j \leq m$ , and

$$f = g_1 \cdot \dots \cdot g_m \quad \text{on } W_1.$$

Now let a neighborhood  $W_2$  of  $\xi$  with  $\overline{W_2} \subseteq W_1$  be given. Then we choose a continuous function  $\chi : X \rightarrow [0, 1]$  with  $\chi = 1$  on  $W_2$  and  $\chi = 0$  on  $X \setminus W_1$ . Then

$$\tilde{f}(\zeta) := \begin{cases} \left( I + \chi(\zeta)(g_1(\zeta) - I) \right) \cdot \dots \cdot \left( I + \chi(\zeta)(g_m(\zeta) - I) \right) & \text{if } \zeta \in W_1, \\ I & \text{if } \zeta \in X \setminus W_1, \end{cases}$$

has the required properties  $\square$

**5.2. Lemma.** *Let  $X$  be a one-dimensional Stein space, and  $A : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$  a holomorphic map. Then  $H^1(X, \mathcal{O}^{\text{GCom}(A)}) = 0$ .*

*Proof.* Let an  $\mathcal{O}^{\text{GCom}(A)}$ -cocycle  $\{f_{ij}\}_{i,j \in I}$  with the covering  $\mathcal{U} = \{U_i\}_{i \in I}$  be given. We have to prove that this cocycle is  $\mathcal{O}^{\text{GCom}(A)}$ -trivial.

Denote by  $S$  the set of non-smooth points of  $X$ . Since  $X$  is one-dimensional,  $S$  is discrete and closed in  $X$ . It follows that  $X$  admits arbitrary fine open coverings such that each point of  $S$  is contained in precisely one of the sets of the covering. Therefore, by Proposition 3.1, we may assume that

$$(5.1) \quad S \cap U_i \cap U_j = \emptyset \quad \text{for all } i, j \in I \text{ with } i \neq j,$$

which implies that, for each  $\xi \in S$ , there is precisely one index in  $I$ ,  $\tau(\xi)$ , such that  $\xi \in U_{\tau(\xi)}$ , and  $\xi \notin U_i$  if  $i \neq \tau(\xi)$ . Shrinking the sets  $U_i$  with  $i \in I \setminus \tau(S)$ , we can moreover achieve that, for each  $\xi \in S$ , there is a neighborhood  $V_\xi$  of  $\xi$  with

$$(5.2) \quad V_\xi \subseteq U_{\tau(\xi)}, \quad \text{and}$$

$$(5.3) \quad V_\xi \cap U_i = \emptyset \quad \text{if } i \neq \tau(\xi).$$

Now let  $\pi : \tilde{X} \rightarrow X$  be the normalization of  $X$  (see, e.g., [L, Ch. VI, §4]). Since, for each  $\xi \in S$ ,  $\pi^{-1}(\xi)$  is finite and  $S$  is discrete and closed in  $X$ , then

$$\tilde{S} := \pi^{-1}(S)$$

is discrete and closed in  $\tilde{X}$ . Further let  $\tilde{A} := A \circ \pi$ ,  $\tilde{U}_i := \pi^{-1}(U_i)$ , and  $\{\tilde{f}_{ij}\}$  the  $\mathcal{O}^{\text{GCom} \tilde{A}}$ -cocycle with the covering  $\{\tilde{U}_i\}_{i \in I}$  defined by

$$\tilde{f}_{ij} := f_{ij} \circ \pi \quad \text{on } \tilde{U}_i \cap \tilde{U}_j.$$

The connected components of  $\tilde{X}$  are the normalizations of the irreducible components of  $X$  (see, e.g., [L, Ch. VI, §4.2]). Since  $X$  is one-dimensional, this implies by the Puiseux theorem (see, e.g., [L, Ch. VI, §4.1]) that the connected components of  $\tilde{X}$  are Riemann surfaces. Since  $X$  is Stein and therefore non of the irreducible components of  $X$  is compact, it follows that each of the connected components



of  $\tilde{X}$  is a non-compact connected Riemann surface. Therefore, by Grauert's theorem [Gr, Satz 7] (see also [F, Theorem 30.4] or [Fc, Theorem 5.3.1]), we have  $H^1(\tilde{X}, \mathcal{O}^{\text{GL}(n, \mathbb{C})}) = 0$ . In particular,

$$(5.4) \quad \tilde{f}_{ij} = \tilde{h}_i \tilde{h}_j^{-1} \quad \text{on} \quad \tilde{U}_i \cap \tilde{U}_j$$

for some family  $\tilde{h}_i \in \mathcal{O}^{\text{GL}(n, \mathbb{C})}(U_i)$ . Since  $\tilde{f}_{ij} \tilde{A} = \tilde{A} \tilde{f}_{ij}$ , it follows that  $\tilde{h}_i^{-1} \tilde{A} \tilde{h}_i = \tilde{h}_j^{-1} \tilde{A} \tilde{h}_j$  on  $\tilde{U}_i \cap \tilde{U}_j$ . Hence, there is a well-defined global holomorphic map  $\tilde{B} : \tilde{X} \rightarrow \text{Mat}(n \times n, \mathbb{C})$  with

$$(5.5) \quad \tilde{B} = \tilde{h}_i^{-1} \tilde{A} \tilde{h}_i \quad \text{on} \quad \tilde{U}_i.$$

Then, by definition of  $\tilde{B}$ ,  $\tilde{B}$  and  $\tilde{A}$  are locally holomorphically similar on  $\tilde{X}$ . Therefore, by Guralnick's result (Theorem 1.2 above), we can find a holomorphic  $\tilde{H} : \tilde{X} \rightarrow \text{GL}(n, \mathbb{C})$  with

$$(5.6) \quad \tilde{H} \tilde{B} \tilde{H}^{-1} = \tilde{A} \quad \text{on} \quad \tilde{X}.$$

Then, by (5.5) and (5.6),  $\tilde{H} \tilde{h}_i^{-1} \tilde{A} \tilde{h}_i \tilde{H}^{-1} = \tilde{H} \tilde{B} \tilde{A} = \tilde{A}$  on  $\tilde{U}_i$ , i.e.,

$$(5.7) \quad \tilde{H} \tilde{h}_i^{-1} \in \mathcal{O}^{\text{GCom } \tilde{A}}(\tilde{U}_i).$$

By (5.2), for each  $\xi \in S$ ,  $\pi^{-1}(V_\xi) \subseteq \tilde{U}_{\tau(\xi)}$ . Since  $\pi^{-1}(V_\xi)$  is a neighborhood of the finite set  $\pi^{-1}(\xi)$  and since, by (5.7),  $\tilde{H} \tilde{h}_{\tau(\xi)}^{-1} \in \mathcal{O}^{\text{GCom } \tilde{A}}(\tilde{U}_{\tau(\xi)})$  if  $\xi \in S$ , this implies by Lemma 5.1, that, for each  $\xi \in S$ , there exist neighborhoods  $\tilde{W}_1(\pi^{-1}(\xi))$  and  $\tilde{W}_2(\pi^{-1}(\xi))$  of  $\pi^{-1}(\xi)$  and a map

$$(5.8) \quad \tilde{C}_\xi \in \widehat{\mathcal{C}}^{\text{GCom } \tilde{A}}(\tilde{X})$$

such that

$$(5.9) \quad \tilde{W}_1(\pi^{-1}(\xi)) \subseteq \pi^{-1}(V_\xi),$$

$$(5.10) \quad \overline{\tilde{W}_2(\pi^{-1}(\xi))} \subseteq \tilde{W}_1(\pi^{-1}(\xi)),$$

$$(5.11) \quad \tilde{C}_\xi = \tilde{H} \tilde{h}_{\tau(\xi)}^{-1} \quad \text{on} \quad \tilde{W}_2(\pi^{-1}(\xi)),$$

$$(5.12) \quad \tilde{C}_\xi = I \quad \text{on} \quad \tilde{X} \setminus \tilde{W}_1(\pi^{-1}(\xi)).$$

Set

$$\tilde{W}_1 = \bigcup_{\xi \in S} \tilde{W}_1(\pi^{-1}(\xi)) \quad \text{and} \quad \tilde{W}_2 = \bigcup_{\xi \in S} \tilde{W}_2(\pi^{-1}(\xi)).$$

By (5.3) and (5.2),  $V_\xi \cap V_\eta = \emptyset$  and, hence,  $\pi^{-1}(V_\xi) \cap \pi^{-1}(V_\eta) = \emptyset$  if  $\xi, \eta \in S$  with  $\xi \neq \eta$ . By (5.9) this implies that  $\tilde{W}_1(\pi^{-1}(\xi)) \cap \tilde{W}_1(\pi^{-1}(\eta)) = \emptyset$  if  $\xi, \eta \in S$  with  $\xi \neq \eta$ . Therefore and by (5.8) and (5.12), there is a well-defined map

$$(5.13) \quad \tilde{C} \in \widehat{\mathcal{C}}^{\text{GCom } \tilde{A}}(\tilde{X})$$

with

$$(5.14) \quad \tilde{C} = \tilde{C}_\xi \quad \text{on} \quad \tilde{W}_2(\pi^{-1}(\xi)), \quad \text{for each} \quad \xi \in S, \quad \text{and}$$

$$(5.15) \quad \tilde{C} = I \quad \text{on} \quad \tilde{X} \setminus \tilde{W}_1.$$

Define  $\tilde{c}_i = \tilde{h}_i \tilde{H}^{-1} \tilde{C}$  on  $\tilde{U}_i$ . Then, by (5.13) and (5.7),

$$(5.16) \quad \tilde{c}_i \in \widehat{\mathcal{C}}^{\text{GCom } \tilde{A}}(\tilde{U}_i) \quad \text{for all} \quad i \in I,$$

and, by (5.4),

$$(5.17) \quad \tilde{f}_{ij} = \tilde{c}_i \tilde{c}_j^{-1} \quad \text{on} \quad \tilde{U}_i \cap \tilde{U}_j.$$

Now it remains to find maps  $c_i \in \widehat{\mathcal{C}}^{\text{GCom } A}(U_i)$ ,  $i \in I$ , with

$$(5.18) \quad \tilde{c}_i = c_i \circ \pi \quad \text{on} \quad U_i.$$

Indeed, since  $\pi$  is biholomorphic from  $\tilde{X} \setminus \tilde{S}$  onto  $X \setminus S$ , then it follows from (5.18) and (5.17) that  $f_{ij} = c_i c_j^{-1}$  on  $U_i \cap U_j$ , i.e.,  $\{f_{ij}\}$  is  $\widehat{\mathcal{C}}^{\text{GCom } A}$ -trivial, which implies by Proposition 4.6 that  $\{f_{ij}\}$  is  $\mathcal{O}^{\text{GCom } A}$ -trivial.

If  $i \in I \setminus \tau(S)$ , then  $U_i \subseteq X \setminus S$ , by (5.3). Therefore, since  $\pi$  is biholomorphic from  $\tilde{X} \setminus \tilde{S}$  onto  $X \setminus S$ , then we can define  $c_i = \tilde{c}_i \circ \pi^{-1}$ .

Let  $\xi \in S$ . Denote by  $X_\xi$  the set of germs of  $X$  at  $\xi$ . By Puiseux's theorem (see, e.g., [L, Ch. VI, §4.1]), for each  $\tilde{\xi} \in \pi^{-1}(\xi)$ ,  $\pi$  is homeomorphic from some neighborhood of  $\tilde{\xi}$  onto a representative of one of the germs from  $X_\xi$ . This implies that there is a neighborhood of  $\xi$  in  $X$ ,  $W_2(\xi)$ , with  $\pi^{-1}(W_2(\xi)) \subseteq \tilde{W}_2(\pi^{-1}(\xi))$ . Therefore, it follows from (5.14) and (5.11) that

$$(5.19) \quad \tilde{c}_{\tau(\xi)} = \tilde{h}_{\tau(\xi)} \tilde{H}^{-1} \tilde{C} = \tilde{h}_{\tau(\xi)} \tilde{H}^{-1} \tilde{C}_\xi = I \quad \text{on} \quad \pi^{-1}(W_2(\xi)) \quad \text{for all} \quad \xi \in S.$$

By (5.3),  $S \cap U_{\tau(\xi)} = \{\xi\}$ . Therefore,  $\pi$  is biholomorphic from  $\tilde{U}_{\tau(\xi)} \setminus \pi^{-1}(\xi)$  onto  $U_{\tau(\xi)} \setminus \{\xi\}$ . By (5.16) and (5.19) this implies that there is a well-defined map  $c_{\tau(\xi)} \in \widehat{\mathcal{C}}^{\text{GCom } A}(U_{\tau(\xi)})$  with  $\tilde{c}_{\tau(\xi)} = c_{\tau(\xi)} \circ \pi$  on  $\tilde{U}_{\tau(\xi)}$ , i.e., we have (5.18) for  $i = \tau(\xi)$ .  $\square$

**5.3. Remark.** Lemma 5.2 contains the statement  $H^1(X, \mathcal{O}^{\text{GCom } \Phi}) = 0$ , for each matrix  $\Phi \in \text{Mat}(n \times n, \mathbb{C})$  and each one-dimensional Stein space  $X$ . Since  $\text{GCom}(\Phi)$  is connected (Lemma 4.2), this is a special case of the statement

$$(5.20) \quad H^1(X, \mathcal{O}^G) = 0,$$

for each connected complex Lie group  $G$  and each one-dimensional Stein space  $X$ . If  $X$  is smooth, (5.20) was proved by H. Grauert [Gr, Satz 7].

For non-smooth  $X$ , surprisingly, it seems that there is no explicit reference for (5.20) in the literature, except for  $G = \text{GL}(n, \mathbb{C})$ , see [Fc, Theorem 7.3.1 (c) or Corollary 7.3.2, 1.]. Therefore I asked colleagues and got two answers.

F. Forstnerič answered that, by [H], each one-dimensional Stein space has the homotopy type of a one-dimensional CW complex and, therefore,

$$(5.21) \quad H^1(X, \mathcal{C}^G) = 0, \quad ^2$$

which then implies (5.20) by Grauert's Oka principle [Gr, Satz I] (see also [Fc, 7.2.1]).

J. Ruppenthal proposed to pass to the normalization of  $X$ , which is smooth. At least if  $X$  is irreducible and, hence, homeomorphic to its normalization, this immediately reduces the topological statement (5.21) to the smooth case, which then implies (5.20), again by Grauert's Oka principle. This idea is used in the proof of Lemma 5.2 above.

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<sup>2</sup>Indeed, let  $f$  be a  $\mathcal{C}_X^G$  cocycle, and let  $B$  be the principal  $G$ -bundle defined by  $f$ . Then (by definition of  $B$ ) the  $\mathcal{C}_X^G$ -triviality of  $f$  (which we have to prove) is equivalent to the existence of a global continuous section of  $B$ , and the existence of such a global continuous section follows, e.g., from [St, Theorem 11.5 and §29.1].

*Proof of Theorem 1.3.* Since  $A$  and  $B$  are locally holomorphically similar at each point of  $X$ , we can find an open covering  $\{U_i\}_{i \in I}$  of  $X$  and holomorphic maps  $H_i : U_i \rightarrow \mathrm{GL}(n, \mathbb{C})$ ,  $i \in I$ , such that

$$(5.22) \quad B = H_i^{-1} A H_i \quad \text{on } U_i.$$

It follows that  $A H_i H_j^{-1} = H_i H_j^{-1} A$  on  $U_i \cap U_j$ . Hence,  $\{H_i H_j^{-1}\}_{i,j \in I}$  is an  $\mathcal{O}^{\mathrm{GCom} A}$ -cocycle. By Lemma 5.2, this cocycle is  $\mathcal{O}^{\mathrm{GCom} A}$ -trivial, i.e.,  $H_i H_j^{-1} = h_i h_j^{-1}$  on  $U_i \cap U_j$ , for some family  $h_i \in \mathcal{O}^{\mathrm{GCom} A}(U_i)$ . Hence  $h_i^{-1} H_i = h_j^{-1} H_j$  on  $U_i \cap U_j$ , which means that there is a well-defined map  $H \in \mathcal{O}^{\mathrm{GCom} A}(X)$  with  $H = i_i^{-1} H_i$  on  $U_i$ . From (5.22) and the relations  $H_i A H_i^{-1} = A$  it follows that  $B = H^{-1} A H$  on  $X$ .  $\square$

## 6. PROOF OF THEOREM 1.5

We show that the statements of this theorem are known or easily follow from known results. First we collect these known results.

We begin with following deep result of K. Spallek, which is a special case of [Sp1, Satz 5.4] (see also the beginning of [Sp2]).

**6.1. Proposition.** *Let  $X$  be a complex space,  $M : X \rightarrow \mathrm{Mat}(n \times m, \mathbb{C})$  holomorphic, and  $\xi \in X$ . Then there exists  $k \in \mathbb{N}$  (depending on  $M$  and  $\xi$ ) such that the following holds.*

*Suppose  $U$  is a neighborhood of  $\xi$  and  $f : U \rightarrow \mathbb{C}^m$  is a  $\mathcal{C}^k$  map with  $Mf = 0$  on  $U$ . Then there exist a neighborhood  $V \subset U$  of  $\xi$  and a holomorphic map  $h : V \rightarrow \mathbb{C}^m$  with  $Mh = 0$  on  $V$  and  $h(\xi) = f(\xi)$ .*

The next proposition is well-known and more easy to prove.

**6.2. Proposition.** *Let  $D$  be a domain in  $\mathbb{C}$ ,  $M : D \rightarrow \mathrm{Mat}(n \times m, \mathbb{C})$  holomorphic,  $M \not\equiv 0$ , and  $\xi \in D$ . Then:*

(i) *There exist an open neighborhood  $U$  of  $\xi$ , holomorphic maps  $E : U \rightarrow \mathrm{GL}(n, \mathbb{C})$ ,  $F : U \rightarrow \mathrm{GL}(m, \mathbb{C})$ , and nonnegative integers  $\kappa_1, \dots, \kappa_r$  such that*

$$(6.1) \quad M(\zeta) = E(\zeta) \begin{pmatrix} \Delta(\zeta) & 0 \\ 0 & 0 \end{pmatrix} F(\zeta) \quad \text{for all } \zeta \in U, \quad ^3$$

*where  $\Delta(\zeta)$  is the diagonal matrix with the diagonal  $(\zeta - \xi)^{\kappa_1}, \dots, (\zeta - \xi)^{\kappa_r}$ .*

(ii) *Let  $W \subseteq D$  be a neighborhood of  $\xi$  and  $c : W \rightarrow \mathbb{C}^m$  a continuous map with*

$$(6.2) \quad M c = 0 \quad \text{on } W.$$

*Then there exist a neighborhood  $V \subseteq W$  of  $\xi$  and a holomorphic map  $h : V \rightarrow \mathbb{C}^m$  with*

$$M h = 0 \quad \text{on } V \quad \text{and} \quad h(\xi) = c(\xi).$$

Part (i) is an application of the Smith factorization theorem (see, e.g., [J, Ch. III, Sect. 8]) to the ring of germs of holomorphic functions in neighborhoods of  $\xi$ . (A direct proof of part (i) can be found, e.g., in [GL, Theorem 4.3.1]).

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<sup>3</sup>Possibly, some of the zeros in this block matrix have to be omitted.

Part (ii) is a corollary of part (i). Indeed, let  $U$ ,  $E$ ,  $F$  and  $r$  be as in part (i), and let  $W$  and  $c$  be as in part (ii). Set

$$\begin{pmatrix} f_1(\zeta) \\ \vdots \\ f_m(\zeta) \end{pmatrix} = F(\zeta)c(\zeta) \quad \text{for } \zeta \in U \cap W.$$

Then, by (6.1) and (6.2),  $f_1(\zeta) = \dots = f_r(\zeta) = 0$  for  $\zeta \in (U \cap W) \setminus \{\xi\}$ , and, hence, by continuity,  $f_1(\xi) = \dots = f_r(\xi) = 0$ . It remains to define

$$h(\zeta) = F(\zeta)^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f_{r+1}(\xi) \\ \vdots \\ f_m(\xi) \end{pmatrix} \quad \text{for } \zeta \in V := U \cap W.$$

Finally, note the following fact, which is nowadays well-known. Proofs can be found, e.g., in [W]<sup>4</sup> or in [Sh, Corollary 2].

**6.3. Proposition.** *Let  $X$  be a complex space,  $M : X \rightarrow \text{Mat}(n \times m, \mathbb{C})$  holomorphic, and  $\xi \in X$  such that the dimension of  $\text{Ker } M(\zeta)$  does not depend on  $\zeta$  in some neighborhood of  $\xi$ . Then there exist a neighborhood  $U$  of  $\xi$  such that the family  $\{\text{Ker } M(\zeta)\}_{\zeta \in U}$  is a holomorphic sub-vector bundle of  $U \times \mathbb{C}^m$ .*

*Proof of Theorem 1.5.* Denote by  $\text{End}(\text{Mat}(n \times n, \mathbb{C}))$  the space of linear endomorphisms of  $\text{Mat}(n \times n, \mathbb{C})$ , and let  $\varphi_{A,B} : X \rightarrow \text{End}(\text{Mat}(n \times n, \mathbb{C}))$  be the holomorphic map defined by

$$\varphi_{A,B}(\zeta)\Phi = A(\zeta)\Phi - \Phi B(\zeta) \quad \text{for } \zeta \in X \text{ and } \Phi \in \text{Mat}(n \times n, \mathbb{C}).$$

Fix a basis of  $\text{Mat}(n \times n, \mathbb{C})$ , and let  $M_{A,B}$  be the representation matrix of  $\varphi_{A,B}$  with respect to this basis.

First assume that condition (i) is satisfied. Then the claim of the theorem was proved by W. Wasow [W]. He considered only the case when  $X$  is a domain in  $\mathbb{C}$ , but his proof works also in the general case. It goes as follows:

By definition of  $\varphi_{A,B}$ , (1.1) is the kernel of  $\varphi_{A,B}(\zeta)$ . Therefore, we have a neighborhood  $U$  of  $\xi$  and a number  $r \in \mathbb{N}$  such that

$$\dim \text{Ker } \varphi(\zeta) = r \quad \text{for all } \zeta \in U.$$

By Proposition 6.3, this means that the family  $\{\text{Ker } \varphi_{A,B}(\zeta)\}_{\zeta \in U}$  is a holomorphic sub-vector bundle of the product bundle  $U \times \text{Mat}(n \times n, \mathbb{C})$ . Since  $\Phi \in \text{Ker } \varphi_{A,B}(\xi)$ , then, after shrinking  $U$ , we can find a holomorphic section  $H$  of this bundle with  $H(\xi) = \Phi$ .  $\square$

If (ii) or (iii) is satisfied, then the claim of the theorem follows immediately from Propositions 6.2 (ii) and 6.1, respectively, with  $M = M_{A,B}$ .  $\square$

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<sup>4</sup>Lemma 6.3 is not explicitly stated in [W], but it follows immediately from Lemma 1 of [W]. Also, in [W],  $X$  is a domain in the complex plane, but the proof given there works also in the general case.

**6.4. Remark.** This proof shows that condition (iii) in Theorem 1.5 can be replaced by the following: There exists a positive integer  $k$  depending on  $\xi$ ,  $A$  and  $B$  such that, if there exist a neighborhood  $U$  of  $\xi$  and a  $\mathcal{C}^k$  map  $T : V \rightarrow \text{Mat}(n \times n, \mathbb{C})$  such that  $T(\xi) = \Phi$  and  $TB = AT$  on  $U$ , then there exist a neighborhood  $V \subseteq U$  of  $\xi$  and a holomorphic map  $H : V \rightarrow \text{Mat}(n \times n, \mathbb{C})$  with  $H(\xi) = \Phi$  and  $HB = AH$  on  $V$ .

## 7. LOCAL COUNTEREXAMPLES

Let  $z$  and  $w$  be the canonical complex coordinate functions on  $\mathbb{C}^2$ .

We begin with the following observation of O. Forster and K. J. Ramspott [FR2, page 159]): If  $\alpha$ ,  $\beta$  and  $\gamma$  are holomorphic functions in a neighborhood of the origin in  $\mathbb{C}^2$ , which solve the equation

$$\alpha z^3 + \beta w^3 + \gamma z^2 w^2 = 0$$

in this neighborhood, then, comparing the coefficients in the Taylor series, it follows easily that  $\alpha(0) = \beta(0) = \gamma(0) = 0$ . With continuous functions however, this equation can be solved with  $\gamma(0) \neq 0$ . For example,

$$\frac{\bar{z}w^2}{|z|^2 + |w|^2} z^3 + \frac{\bar{w}z^2}{|z|^2 + |w|^2} w^3 = z^2 w^2.$$

We use a  $\mathcal{C}^\ell$ -version of this.

**7.1.** Let  $\mathbb{B}^2$  be the open unit ball in  $\mathbb{C}^2$ ,  $\ell \in \mathbb{N}$ ,

$$A = \begin{pmatrix} z^{2+\ell} w^{2+\ell} & z^{3+\ell} \\ w^{3+\ell} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & z^{3+\ell} \\ w^{3+\ell} & z^{2+\ell} w^{2+\ell} \end{pmatrix},$$

$$c_z = \frac{\bar{z}w^{2+\ell}}{|z|^2 + |w|^2}, \quad c_w = \frac{\bar{w}z^{2+\ell}}{|z|^2 + |w|^2}, \quad S = \begin{pmatrix} 1 & c_w \\ -c_z & 1 \end{pmatrix}.$$

Then it is again easy to see that

$$(7.1) \quad c_z z^{3+\ell} + c_w w^{3+\ell} = z^{2+\ell} w^{2+\ell} \quad \text{on } \mathbb{B}^2,$$

and, comparing the coefficients of the Taylor series<sup>5</sup>, we get

**7.2. Lemma.** *Suppose  $\alpha$ ,  $\beta$ ,  $\gamma$  are holomorphic functions in a neighborhood of the origin in  $\mathbb{C}^2$  such that*

$$\alpha z^{\ell+3} + \beta w^{\ell+3} + \gamma z^{\ell+2} w^{\ell+2} = 0$$

*in this neighborhood. Then  $\alpha(0) = \beta(0) = \gamma(0) = 0$ .*

Also it is easy to see that the functions  $c_z$  and  $c_w$  are of class  $\mathcal{C}^\ell$  on  $\mathbb{C}^2$  and that  $|c_z c_w| < 1$  on  $\mathbb{B}^2$ . Hence  $S$  is of class  $\mathcal{C}^\ell$  on  $\mathbb{C}^2$ , and  $S(\zeta) \in \text{GL}(n, \mathbb{C})$  for all  $\zeta \in \mathbb{B}^2$ . Moreover,

$$AS = \begin{pmatrix} z^{2+\ell} w^{2+\ell} - c_z z^{3+\ell} & c_w z^{\ell+2} w^{\ell+2} + z^{3+\ell} \\ w^{3+\ell} & c_w w^{3+\ell} \end{pmatrix},$$

$$SB = \begin{pmatrix} c_w w^{3+\ell} & z^{3+\ell} + c_w z^{2+\ell} w^{2+\ell} \\ w^{3+\ell} & -c_z z^{3+\ell} + z^{2+\ell} w^{2+\ell} \end{pmatrix},$$

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<sup>5</sup>Below we explain this in detail in the case of Lemmas 7.5 and 7.6, each of which is stronger than Lemma 7.2.

which implies by (7.1) that

$$(7.2) \quad SBS^{-1} = A \quad \text{on} \quad \mathbb{B}^2.$$

Hence  $A$  and  $B$  are globally  $\mathcal{C}^\ell$  similar on  $\mathbb{B}^2$ . On the other hand, we have

**7.3. Lemma.** *Let  $U$  be an open neighborhood of the origin in  $\mathbb{C}^2$ , and  $H : U \rightarrow \text{Mat}(2 \times 2, \mathbb{C})$  holomorphic. Then:*

- (i) *If, on  $U$ ,  $AH = HB$  or  $HA = BH$ , then  $H(0) = 0$ .*
- (ii) *If, on  $U$ ,  $AH = HA$  or  $AB = BA$ , then  $H(0) = \lambda I_2$  for some  $\lambda \in \mathbb{C}$ .*

*Proof.* Let  $H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Then

$$(7.3) \quad AH = \begin{pmatrix} az^{2+\ell}w^{2+\ell} + cz^{3+\ell} & bz^{2+\ell}w^{2+\ell} + dz^{3+\ell} \\ aw^{3+\ell} & bw^{3+\ell} \end{pmatrix},$$

$$(7.4) \quad HB = \begin{pmatrix} bw^{3+\ell} & az^{3+\ell} + bz^{2+\ell}w^{2+\ell} \\ dw^{3+\ell} & cz^{3+\ell} + dz^{2+\ell}w^{2+\ell} \end{pmatrix},$$

$$(7.5) \quad HA = \begin{pmatrix} az^{2+\ell}w^{2+\ell} + bw^{3+\ell} & az^{3+\ell} \\ cz^{2+\ell}w^{2+\ell} + dw^{3+\ell} & cz^{3+\ell} \end{pmatrix},$$

$$(7.6) \quad BH = \begin{pmatrix} cz^{3+\ell} & dz^{3+\ell} \\ aw^{3+\ell} + cz^{2+\ell}w^{2+\ell} & bw^{3+\ell} + dz^{2+\ell}w^{2+\ell} \end{pmatrix}.$$

In particular:

- if  $AH = HB$ , then  $az^{2+\ell}w^{2+\ell} + cz^{3+\ell} = bw^{3+\ell} = cz^{3+\ell} + dz^{2+\ell}w^{2+\ell}$ ,
- if  $HA = BH$ , then  $az^{2+\ell}w^{2+\ell} + bw^{3+\ell} = cz^{3+\ell} = bw^{3+\ell} + dz^{2+\ell}w^{2+\ell}$ ,
- if  $AH = HA$ , then  $cz^{3+\ell} = bw^{3+\ell}$  and  $(a-d)z^{3+\ell} = bz^{2+\ell}w^{2+\ell}$ ,
- if  $BH = HB$ , then  $bw^{3+\ell} = cz^{3+\ell}$  and  $(d-a)w^{3+\ell} = cz^{2+\ell}w^{2+\ell}$ .

By Lemma 7.2, this yields:

- if  $AH = HB$  or  $HA = BH$ , then  $a(0) = b(0) = c(0) = d(0) = 0$ ,
- if  $AH = HA$  or  $BH = HB$ , then  $b(0) = c(0) = 0$  and  $a(0) = d(0)$ .  $\square$

Lemma 7.3 (i) in particular says that  $A$  and  $B$  are not locally holomorphically similar at 0. At the end of this section we prove the following stronger

**7.4. Theorem.** *Suppose (a)  $X = \{z^p = w^q\}$ , where  $p, q \in \mathbb{N}$  such that  $\ell + 2 < q < p$  and  $p, q$  are relatively prime, or (b)  $X$  is the union of  $2\ell + 5$  pairwise different one-dimensional linear subspaces of  $\mathbb{C}^2$ .*

*Then the restrictions  $A|_X$  and  $B|_X$  are not locally holomorphically similar at 0.*

**7.5. Lemma.** *Let  $X = \{z^p = w^q\}$ , where  $p, q \in \mathbb{N}$  such that  $\ell + 2 < q < p$  and  $p, q$  are relatively prime. Suppose  $U$  is a neighborhood of the origin in  $\mathbb{C}^2$ , and  $\alpha, \beta, \gamma : U \rightarrow \mathbb{C}$  are holomorphic such that*

$$(7.7) \quad \alpha z^{\ell+3} + \beta w^{\ell+3} + \gamma z^{\ell+2}w^{\ell+2} = 0 \quad \text{on} \quad X \cap U.$$

*Then  $\alpha(0) = \beta(0) = \gamma(0) = 0$ .*

*Proof.* Choose  $0 < \varepsilon < 1$  so small that the closed bidisk  $\max(|z|, |w|) \leq \varepsilon$  is contained in  $U$ , and let

$$\sum_{j,k=0}^{\infty} \alpha_{jk} z^j w^k, \quad \sum_{j,k=0}^{\infty} \beta_{jk} z^j w^k, \quad \sum_{j,k=0}^{\infty} \gamma_{jk} z^j w^k$$

be the Taylor series of  $\alpha$ ,  $\beta$  and  $\gamma$ , respectively. Then, by (7.7),

$$\sum_{j,k=0}^{\infty} \alpha_{jk} z^{j+\ell+3} w^k + \sum_{j,k=0}^{\infty} \beta_{jk} z^j w^{k+\ell+3} + \sum_{j,k=0}^{\infty} \gamma_{jk} z^{j+\ell+2} w^{k+\ell+2} = 0$$

if  $z^p = w^q$  and  $\max(|z|, |w|) < \varepsilon$ . With  $z = t^q$  and  $w = t^p$  for  $0 \leq t < \varepsilon$ , this yields

$$\sum_{j,k=0}^{\infty} \alpha_{jk} t^{(j+\ell+3)q+kp} + \sum_{j,k=0}^{\infty} \beta_{jk} t^{jq+(k+\ell+3)p} + \sum_{j,k=0}^{\infty} \gamma_{jk} t^{(j+\ell+2)q+(k+\ell+2)p} = 0$$

for all  $0 \leq t < \varepsilon$ . Comparing the coefficients of  $t^{(\ell+3)q}$ ,  $t^{(\ell+3)p}$  and  $t^{(\ell+2)(p+q)}$ , we get

$$\begin{aligned} \alpha_{00} + \sum_{(j,k) \in A_\beta} \beta_{jk} + \sum_{(j,k) \in A_\gamma} \gamma_{jk} &= 0, \\ \sum_{(j,k) \in B_\alpha} \alpha_{jk} + \beta_{00} + \sum_{(j,k) \in B_\gamma} \gamma_{jk} &= 0, \\ \sum_{(j,k) \in C_\alpha} \alpha_{jk} + \sum_{(j,k) \in C_\beta} \beta_{jk} + \gamma_{00} &= 0, \end{aligned}$$

where  $A_\beta, \dots, C_\beta$  are the subsets of  $\mathbb{N} \times \mathbb{N}$  defined by

$$\begin{aligned} (j, k) \in A_\beta &\stackrel{\text{def}}{\iff} jq + (k + \ell + 3)p = (\ell + 3)q \iff (k + \ell + 3)p = (\ell + 3 - j)q, \\ (j, k) \in A_\gamma &\stackrel{\text{def}}{\iff} (j + \ell + 2)q + (k + \ell + 2)p = (\ell + 3)q \iff (k + \ell + 2)p = (1 - j)q, \\ (j, k) \in B_\alpha &\stackrel{\text{def}}{\iff} (j + \ell + 3)q + kp = (\ell + 3)p \iff (j + \ell + 3)q = (\ell + 3 - k)p, \\ (j, k) \in B_\gamma &\stackrel{\text{def}}{\iff} (j + \ell + 2)q + (k + \ell + 2)p = (\ell + 3)p \iff (j + \ell + 2)q = (1 - k)p, \\ (j, k) \in C_\alpha &\stackrel{\text{def}}{\iff} (j + \ell + 3)q + kp = (\ell + 2)(p + q) \iff (j + 1)q = (\ell + 2 - k)p, \\ (j, k) \in C_\beta &\stackrel{\text{def}}{\iff} jq + (k + \ell + 3)p = (\ell + 2)(p + q) \iff (\ell + 2 - j)q = (k + 1)p. \end{aligned}$$

It is sufficient to prove that  $A_\beta = A_\gamma = B_\alpha = B_\gamma = C_\alpha = C_\beta = \emptyset$ .

Assume  $(k + \ell + 3)p = (\ell + 3 - j)q$ . Contrary to  $q < p$ , then it follows

$$p = \frac{\ell + 3 - j}{k + \ell + 3}q \leq \frac{\ell + 3}{k + \ell + 3} \leq q.$$

Assume  $(k + \ell + 2)p = (1 - j)q$ . Contrary to  $p > p/2$ , then it follows

$$p = \frac{1 - j}{k + \ell + 2}q \leq \frac{q}{2} < \frac{p}{2}.$$

Assume  $(j + \ell + 3)q = (\ell + 3 - k)p$ . Since  $p$  and  $q$  are relatively prime, this implies that  $j + \ell + 3 = np$ , for some integer  $n \in \mathbb{N}^*$ .  $n = 1$  is not possible, for this would imply that  $p = j + \ell + 3 \leq \ell + 3 \leq q < p$ .  $n \geq 2$  is also impossible, as this would imply that  $p \geq \ell + 3 \geq j + \ell + 3 \geq 2p$ .

Assume  $(j + \ell + 2)q = (1 - k)p$ . This implies that  $k = 0$  and therefore  $(j + \ell + 2)q = p$ , which is not possible, since  $p$  and  $q$  are relatively prime.

Assume  $(j + 1)q = (\ell + 2 - k)p$ . As  $p$  and  $q$  are relatively prime, this implies that  $\ell + 2 - k$  is positive and can be divided by  $q$ . In particular,  $\ell + 2 - k \geq q$ , which is not possible, for  $\ell + 2 - k < q$ .

Assume  $(\ell + 2 - j)q = (k + 1)p$ . Since  $p$  and  $q$  are relatively prime, this implies that  $\ell + 2 - j = np$  for some  $n \in \mathbb{N}^*$  and, further,  $p > \ell + 2 \geq \ell + 2 - j = np \geq p$ , contrary to  $q < p$ .  $\square$

**7.6. Lemma.** *Let  $t_1, \dots, t_{2\ell+5}$  be pairwise different complex numbers, and*

$$X := \bigcup_{j=1}^{2\ell+5} \{w = t_j z\}.$$

*Suppose  $U$  is a neighborhood of the origin in  $\mathbb{C}^2$ , and  $\alpha, \beta, \gamma : U \rightarrow \mathbb{C}$  are holomorphic such that*

$$(7.8) \quad \alpha z^{\ell+3} + \beta w^{\ell+3} + \gamma z^{\ell+2} w^{\ell+2} = 0 \quad \text{on } X \cap U.$$

*Then  $\alpha(0) = \beta(0) = \gamma(0) = 0$ .*

*Proof.* To prove that  $\alpha(0) = 0$ , we assume that  $\alpha(0) \neq 0$ . Setting  $b = \beta/\alpha$  and  $c = \gamma/\alpha$ , then we get holomorphic functions  $b, c$  in a neighborhood  $V \subseteq U$  of 0 such that

$$z^{\ell+3} = c z^{2+\ell} w^{\ell+2} - b w^{\ell+3} = 0 \quad \text{on } X \cap V.$$

It follows that, for  $1 \leq j \leq 2\ell + 5$  and all  $\zeta$  in some neighborhood of zero in the complex plane,

$$\zeta^{\ell+3} = c(\zeta, t_j \zeta) \zeta^{2\ell+4} t_j^{\ell+2} - b(\zeta, t_j \zeta) \zeta^{\ell+3} t_j^{\ell+3}$$

and, hence,

$$1 = c(\zeta, t_j \zeta) \zeta^{\ell+1} t_j^{\ell+2} - b(\zeta, t_j \zeta) t_j^{\ell+3}.$$

Hence,  $1 = -t_j^{\ell+3} \beta(0, 0)$  for  $1 \leq j \leq 2\ell + 5$ . This implies that  $\beta(0, 0) \neq 0$  and  $t_1, \dots, t_{2\ell+5}$  are solutions of the equation

$$t^{\ell+3} = -\frac{1}{\beta(0, 0)}.$$

As  $2\ell + 5 > \ell + 3$  and the numbers  $t_j$  are pairwise different, this is impossible.

Changing the roles of  $z$  and  $w$ , one proves in the same way that  $\beta(0) = 0$ .

Finally we assume that  $\gamma(0) \neq 0$ . Setting  $a = \alpha/\gamma$  and  $b = \beta/\gamma$ , then we get holomorphic functions  $a, b$  in a neighborhood  $V \subseteq U$  of 0 such that

$$a z^{\ell+3} + b w^{\ell+3} = z^{\ell+2} w^{\ell+2} \quad \text{on } X \cap V.$$

It follows that, for  $1 \leq j \leq 2\ell + 5$  and all  $\zeta$  in some neighborhood  $\Omega$  of zero in the complex plane

$$a(\zeta, t_j \zeta) \zeta^{\ell+3} + b(\zeta, t_j \zeta) \zeta^{\ell+3} t_j^{\ell+3} = \zeta^{2\ell+4} t_j^{\ell+2}$$

and, hence,

$$a(\zeta, t_j \zeta) + b(\zeta, t_j \zeta) t_j^{\ell+3} = \zeta^{\ell+1} t_j^{\ell+2}.$$

If  $\sum a_{\mu\nu} z^\mu w^\nu$  and  $\sum b_{\mu\nu} z^\mu w^\nu$  are the Taylor series at the origin of  $a$  and  $b$ , respectively, this means that

$$\sum_{\mu, \nu=0}^{\infty} (a_{\mu\nu} t_j^\nu + b_{\mu\nu} t_j^{\nu+\ell+3}) \zeta^{\mu+\nu} = \zeta^{\ell+1} t_j^{\ell+2}$$

for all  $1 \leq j \leq 2\ell + 5$  and  $\zeta \in \Omega$ . Comparing the coefficients of  $\zeta^{\ell+1}$ , this yields

$$\sum_{\mu+\nu=\ell+1} a_{\mu\nu} t_j^\nu + \sum_{\mu+\nu=\ell+1} b_{\mu\nu} t_j^{\nu+\ell+3} = t_j^{\ell+2}, \quad 1 \leq j \leq 2\ell + 5.$$



i.e.,

$$\sum_{\nu=0}^{\ell+1} \alpha_{\ell+1-\nu, \nu} t_j^\nu + \sum_{\nu=\ell+3}^{2\ell+4} \beta_{2\ell+4-\nu, \nu-\ell-3} t_j^\nu = t_j^{\ell+2}, \quad 1 \leq j \leq 2\ell+5.$$

Hence, the system of  $2\ell+5$  linear equations in  $2\ell+5$  variables

$$\sum_{\nu=0}^{2\ell+4} t_j^\nu x_\nu = 0, \quad 1 \leq j \leq 2\ell+5$$

has a non-trivial solution, namely one with  $x_{\ell+2} = -1$ . This not possible, as

$$\det \begin{pmatrix} 1 & t_1 & t_1^2 & \dots & t_1^{2\ell+4} \\ & \dots & \dots & \dots & \dots \\ 1 & t_{2\ell+5} & t_{2\ell+5}^2 & \dots & t_{2\ell+5}^{2\ell+4} \end{pmatrix} = \prod_{1 \leq i < j \leq 2\ell+5} (t_i - t_j) \neq 0.$$

□

*Proof of Theorem 7.4.* Assume there exist a neighborhood  $U$  of the origin in  $\mathbb{C}^2$  and a holomorphic map  $H : U \rightarrow \text{GL}(2, \mathbb{C})$  such that  $H^{-1}AH = B$  on  $X \cap U$ . If  $H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , then, by (7.3) and (7.4), in particular, it follows that

$$az^{2+\ell}w^{2+\ell} + cz^{3+\ell} = bw^{3+\ell} = cz^{3+\ell} + dz^{2+\ell}w^{2+\ell} \quad \text{on } X \cap U,$$

which implies by Lemmas 7.5 and 7.6 that  $a(0) = b(0) = c(0) = d(0) = 0$ , i.e.,  $H(0) = 0$ , which contradicts the assumption that  $H(0)$  is invertible. □

## 8. A GLOBAL COUNTEREXAMPLE

Let  $v_1, v_2, v_3$  denote the canonical complex coordinate functions on  $\mathbb{C}^3$ , and let  $x_j = \text{Re } v_j$  and  $y_j = \text{Im } v_j$ . Set

$$h = v_1 + iv_2, \quad h^* = v_1 - iv_2,$$

$$\mathbb{S}^2 = \{y_1 = y_2 = y_3 = 0\} \cap \{x_1^2 + x_2^2 + x_3^2 = 1\}, \quad \mathbb{S}^1 = \mathbb{S}^2 \cap \{x_3 = 0\}.$$

Then  $hh^* = x_1^2 + x_2^2 = 1$  on  $\mathbb{S}^1$ . Therefore we can find a neighborhood  $N(\mathbb{S}^2)$  in  $\mathbb{C}^3$  of  $\mathbb{S}^2$  and  $\varepsilon > 0$  such that

$$(8.1) \quad |hh^* - 1| < \frac{1}{2} \quad \text{on } N(\mathbb{S}^2) \cap \{-2\varepsilon < x_3 < 2\varepsilon\}.$$

Set

$$\rho = (x_1^2 + x_2^2 + x_3^2 - 1)^3 + y_1^2 + y_2^2 + y_3^2.$$

Then  $\mathbb{S}^2 = \{\rho = 0\}$  and, making  $\varepsilon$  smaller, we can achieve that

$$\mathbb{S}_\varepsilon^2 = \{\rho < \varepsilon\} \subseteq N(\mathbb{S}^2).$$

Moreover, we can choose  $\varepsilon$  so small that  $\rho$  is strictly plurisubharmonic in  $\mathbb{S}_\varepsilon^2$ . Then  $\mathbb{S}_\varepsilon^2$  is Stein. Set

$$U_+ = \mathbb{S}_\varepsilon^2 \cap \{x_3 > -\varepsilon\} \quad \text{and} \quad U_- = \mathbb{S}_\varepsilon^2 \cap \{x_3 < \varepsilon\}.$$

**8.1. Lemma.** (i) *There exist holomorphic  $a_\pm, b_\pm, c_\pm, d_\pm : U_\pm \rightarrow \mathbb{C}$  such that*

$$(8.2) \quad \begin{pmatrix} a_\pm(\zeta) & b_\pm(\zeta) \\ c_\pm(\zeta) & d_\pm(\zeta) \end{pmatrix} \in \text{GL}(2, \mathbb{C}) \quad \text{for all } \zeta \in U_\pm,$$

$$(8.3) \quad \begin{pmatrix} h & 0 \\ 0 & h^* \end{pmatrix} = \begin{pmatrix} a_+ & b_+ \\ c_+ & d_+ \end{pmatrix} \begin{pmatrix} a_- & b_- \\ c_- & d_- \end{pmatrix}^{-1} \quad \text{on } U_+ \cap U_-.$$

(ii) *There do not exist continuous functions  $f_{\pm} : U_{\pm} \rightarrow \mathbb{C}^*$  such that*

$$(8.4) \quad h = \frac{f_+}{f_-} \quad \text{on} \quad U_+ \cap U_-.$$

*Proof.* (i) Since  $\mathbb{S}_{\varepsilon}^2$  is Stein and  $\mathbb{S}_{\varepsilon}^2 = U_+ \cup U_-$ , by Grauert's Oka principle [Satz I][Gr], [Theorem 5.3.1 (ii)][Fc], it is sufficient to find a continuous  $C_+ : U_+ \rightarrow \text{GL}(2, \mathbb{C})$  with

$$(8.5) \quad \begin{pmatrix} h & 0 \\ 0 & h^* \end{pmatrix} = C_+ \quad \text{on} \quad U_+ \cap U_-,$$

which can be done as follows: Take a continuous function  $\chi : \mathbb{R} \rightarrow [0, 1]$  such that  $\chi(t) = 1$  if  $t \leq \varepsilon$  and  $\chi(t) = 0$  if  $t \geq 2\varepsilon$ , and define

$$C_+(\zeta) = \begin{pmatrix} \chi(x_3(\zeta))h(\zeta) & 1 - \chi(x_3(\zeta)) \\ \chi(x_3(\zeta)) - 1 & \chi(x_3(\zeta))h^*(\zeta) \end{pmatrix} \quad \text{for} \quad \zeta \in U_+.$$

If  $\zeta \in U_+ \cap U_-$ , then  $-\varepsilon < x_3(\zeta) < \varepsilon$  and therefore  $\chi(x_3(\zeta)) = 1$ , which implies (8.5). It remains to prove that  $\det C_+(\zeta) \neq 0$  for all  $\zeta \in U_+$ . If  $\zeta \in U_+$  with  $x_3(\zeta) < 2\varepsilon$ , then, by (8.1),  $\text{Re}(h(\zeta)h^*(\zeta)) > 1/2$ , which yields

$$\text{Re} \det C_+(\zeta) \geq \frac{1}{2} \left( \chi(x_3(\zeta)) \right)^2 + \left( 1 - \chi(x_3(\zeta)) \right)^2 \geq \frac{1}{2}.$$

If  $\zeta \in U_+$  with  $x_3(\zeta) \geq 2\varepsilon$ , then  $\chi(x_3(\zeta)) = 0$  and, hence,  $\det C_+(\zeta) = 1$ .

(ii) Assume such functions exist. Then, for  $0 \leq s \leq 1$ , we have continuous closed curves  $\gamma_s^+ : [0, 2\pi] \rightarrow \mathbb{C}^*$  and  $\gamma_s^- : [0, 2\pi] \rightarrow \mathbb{C}^*$ , well-defined by

$$\begin{aligned} \gamma_s^+(t) &= f_+ \left( (\sqrt{1-s^2} \cos t, \sqrt{1-s^2} \sin t, s) \right), \\ \gamma_s^-(t) &= f_- \left( (\sqrt{1-s^2} \cos t, \sqrt{1-s^2} \sin t, -s) \right). \end{aligned}$$

Let

$$\text{Ind} \gamma_s^{\pm} := \frac{1}{2\pi} \int_0^{2\pi} \frac{(\gamma_s^{\pm})'(t)}{\gamma_s^{\pm}(t)} dt$$

be the winding number of  $\gamma_s^{\pm}$ . It is clear that  $\text{Ind} \gamma_s^{\pm}$  depends continuously on  $s$ , and it is well known that  $\text{Ind} \gamma_s^{\pm}$  is always an integer. Therefore

$$\text{Ind} \gamma_1^+ = \text{Ind} \gamma_0^+ \quad \text{and} \quad \text{Ind} \gamma_1^- = \text{Ind} \gamma_0^-.$$

Since  $\gamma_1^+$  and  $\gamma_1^-$  are constant, it follows that  $\text{Ind}(\gamma_0^+) = \text{Ind}(\gamma_0^-) = 0$  and, hence,

$$(8.6) \quad \text{Ind} \frac{\gamma_0^+}{\gamma_0^-} = 0.$$

By definition of  $h$ ,  $h(\cos t, \sin t, 0) = \cos t + i \sin t = e^{it}$ . By (8.4) this yields

$$e^{it} = \frac{f_+(\cos t, \sin t, 0)}{f_-(\cos t, \sin t, 0)} = \frac{\gamma_0^+(t)}{\gamma_0^-(t)}, \quad 0 \leq t \leq 2\pi,$$

which contradicts (8.6).  $\square$

Now, using also the notations introduced in Section 7.1, we set

$$X = \mathbb{S}_{\varepsilon}^2 \times \mathbb{B}^2, \quad X_{\pm} = U_{\pm} \times \mathbb{B}^2,$$

and, for  $(\zeta, \eta) \in X$ ,

$$\tilde{A}(\zeta, \eta) = A(\eta), \quad \tilde{B}(\zeta, \eta) = B(\eta), \quad \tilde{h}(\zeta, \eta) = h(\zeta), \quad \tilde{h}^*(\zeta, \eta) = h^*(\zeta).$$

Further, let  $a_{\pm}, b_{\pm}, c_{\pm}, d_{\pm}$  be as in Lemma 8.1 (i), and define holomorphic maps  $\Theta_{\pm} : X_{\pm} \rightarrow \text{Mat}(4 \times 4, \mathbb{C})$  by the block matrices

$$\Theta_{\pm}(\zeta, \eta) = \begin{pmatrix} a_{\pm}(\zeta)I_2 & b_{\pm}(\zeta)I_2 \\ c_{\pm}(\zeta)I_2 & d_{\pm}(\zeta)I_2 \end{pmatrix}, \quad (\zeta, \eta) \in X_{\pm}.$$

Then, by (8.2) and (8.3),  $\Theta_{\pm}(\zeta, \eta) \in \text{GL}(4, \mathbb{C})$  for all  $(\zeta, \eta) \in X_{\pm}$ , and

$$(8.7) \quad \begin{pmatrix} \tilde{h}I_2 & 0 \\ 0 & \tilde{h}^*I_2 \end{pmatrix} = \Theta_+ \Theta_-^{-1} \quad \text{on} \quad X_+ \cap X_-.$$

Since, obviously,

$$\begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix} \begin{pmatrix} \tilde{h}I_2 & 0 \\ 0 & \tilde{h}^*I_2 \end{pmatrix} = \begin{pmatrix} \tilde{h}I_2 & 0 \\ 0 & \tilde{h}^*I_2 \end{pmatrix} \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix} \quad \text{on} \quad X,$$

this implies that

$$(8.8) \quad \Theta_+ \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix} \Theta_+^{-1} = \Theta_- \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix} \Theta_-^{-1} \quad \text{on} \quad X_+ \cap X_-.$$

Let  $\Phi : X \rightarrow \text{Mat}(4 \times 4, \mathbb{C})$  be defined by the two sides of (8.8).

**8.2. Theorem.**  $\Phi$  and  $\begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix}$  are

- (a) locally holomorphically similar on  $X$ ,
- (b) globally  $\mathcal{C}^{\ell}$  similar on  $X$ ,
- (c) not globally  $\mathcal{C}^{\infty}$  similar on  $X$ .

*Proof.* The local holomorphic similarity is clear by definition of  $\Phi$ .

To prove (b), let  $S$  be as in Section 7.1 and  $\tilde{S}(\zeta, \eta) := S(\eta)$  for  $(\zeta, \eta) \in X$ . Since  $a_{\pm}(\zeta)I_2, b_{\pm}(\zeta)I_2, c_{\pm}(\zeta)I_2$  and  $d_{\pm}(\zeta)I_2$  commute with  $A(\eta)$ , we have

$$(8.9) \quad \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{A} \end{pmatrix} \Theta_{\pm} = \Theta_{\pm} \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{A} \end{pmatrix} \quad \text{on} \quad U_{\pm}.$$

Moreover, it is clear that  $\tilde{S}\tilde{h}^*I_2 = \tilde{h}^*I_2\tilde{S}$  and therefore

$$\begin{pmatrix} I_2 & 0 \\ 0 & \tilde{S} \end{pmatrix} \begin{pmatrix} \tilde{h}I_2 & 0 \\ 0 & \tilde{h}^*I_2 \end{pmatrix} = \begin{pmatrix} \tilde{h}I_2 & 0 \\ 0 & \tilde{h}^*I_2 \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & \tilde{S} \end{pmatrix} \quad \text{on} \quad X,$$

which implies by (8.7) that

$$\Theta_+^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & \tilde{S} \end{pmatrix} \Theta_+ = \Theta_-^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & \tilde{S} \end{pmatrix} \Theta_- \quad \text{on} \quad X_+ \cap X_-$$

and further

$$\begin{pmatrix} I_2 & 0 \\ 0 & \tilde{S}^{-1} \end{pmatrix} \Theta_+^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & \tilde{S} \end{pmatrix} \Theta_+ = \begin{pmatrix} I_2 & 0 \\ 0 & \tilde{S}^{-1} \end{pmatrix} \Theta_-^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & \tilde{S} \end{pmatrix} \Theta_- \quad \text{on} \quad X_+ \cap X_-.$$

Let  $\Psi : X \rightarrow \text{GL}(4, \mathbb{C})$  be the  $\mathcal{C}^{\ell}$  map defined by the two sides of the last equality. Then, by (7.2),

$$\Psi^{-1} \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix} \Psi = \Theta_{\pm}^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & \tilde{S}^{-1} \end{pmatrix} \Theta_{\pm} \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{A} \end{pmatrix} \Theta_{\pm}^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & \tilde{S} \end{pmatrix} \Theta_{\pm} \quad \text{on} \quad X_{\pm}.$$

In view of (8.9), this implies that

$$\Psi^{-1} \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix} \Psi = \Theta_{\pm}^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & \tilde{S}^{-1} \end{pmatrix} \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{A} \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & \tilde{S} \end{pmatrix} \Theta_{\pm} \quad \text{on } X_{\pm},$$

and further, again by (7.2),

$$\Psi^{-1} \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix} \Psi = \Theta_{\pm}^{-1} \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix} \Theta_{\pm} \quad \text{on } X_{\pm}.$$

By definition of  $\Phi$ , this means that  $\Psi^{-1} \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{A} \end{pmatrix} \Psi = \Phi$  on  $X$ , which completes the proof of (b).

To prove (c), we assume that  $\Phi$  and  $\begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix}$  are globally  $\mathcal{C}^{\infty}$  similar on  $X$ . Since  $X$  is Stein, then, by Theorem ??,  $\Phi$  and  $\begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix}$  are even globally holomorphically similar on  $X$ , i.e., we have a holomorphic map  $\Theta : X \rightarrow \text{GL}(4, \mathbb{C})$  with

$$\Theta^{-1} \Phi \Theta = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix} \quad \text{on } X.$$

By definition of  $\Phi$  this means that

$$\Theta^{-1} \Theta_{\pm} \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix} \Theta_{\pm}^{-1} \Theta = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix} \quad \text{on } X_{\pm},$$

i.e.,

$$\Theta^{-1} \Theta_{\pm} \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix} = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix} \Theta^{-1} \Theta_{\pm} \quad \text{on } X_{\pm}.$$

If  $C_{\pm}, D_{\pm}, E_{\pm}, F_{\pm}$  are the  $2 \times 2$  matrices with

$$\Theta^{-1} \Theta_{\pm} = \begin{pmatrix} C_{\pm} & D_{\pm} \\ E_{\pm} & F_{\pm} \end{pmatrix},$$

then this means that, on  $X_{\pm}$ ,

$$C_{\pm} \tilde{A} = \tilde{A} C_{\pm}, \quad F_{\pm} \tilde{B} = \tilde{B} F_{\pm}, \quad E_{\pm} \tilde{A} = \tilde{B} E_{\pm}, \quad D_{\pm} \tilde{B} = \tilde{A} D_{\pm},$$

i.e., for each fixed  $\zeta \in U_{\pm}$ , we have, on  $\mathbb{B}^2$ ,

$$\begin{aligned} C_{\pm}(\zeta, \cdot)A &= AC_{\pm}(\zeta, \cdot), & F_{\pm}(\zeta, \cdot)B &= BF_{\pm}(\zeta, \cdot), \\ E_{\pm}(\zeta, \cdot)A &= BE_{\pm}(\zeta, \cdot), & D_{\pm}(\zeta, \cdot)B &= AD_{\pm}(\zeta, \cdot). \end{aligned}$$

By Lemma 7.3 this yields that, for each  $\zeta \in U_{\pm}$ , there exist  $\gamma_{\pm}(\zeta), \varphi_{\pm}(\zeta) \in \mathbb{C}$  with

$$(8.10) \quad \Theta^{-1}(\zeta, 0) \Theta_{\pm}(\zeta, 0) = \begin{pmatrix} \gamma_{\pm}(\zeta) I_2 & 0 \\ 0 & \varphi_{\pm}(\zeta) I_2 \end{pmatrix} \quad \text{for all } \zeta \in U_{\pm}.$$

Since the maps  $\Theta^{-1} \Theta_{\pm}$  are holomorphic and have invertible values on  $X_{\pm}$ , the so defined functions  $\gamma_{\pm}$  and  $\varphi_{\pm}$  must be holomorphic and different from zero on  $U_{\pm}$ . Moreover, by (8.7), it follows from the equations (8.10) that, for  $\zeta \in U_{+} \cap U_{-}$ ,

$$\Theta(\zeta, 0)^{-1} \begin{pmatrix} h(\zeta) I_2 & 0 \\ 0 & h^{*}(\zeta) I_2 \end{pmatrix} \Theta(\zeta, 0) = \begin{pmatrix} \gamma_{+}(\zeta, 0) \gamma_{-}(\zeta, 0)^{-1} I_2 & 0 \\ 0 & \varphi_{+}(\zeta) \varphi_{-}(\zeta, 0)^{-1} I_2 \end{pmatrix}.$$

In particular, for each  $\zeta \in U_{+} \cap U_{+}$ , the matrices

$$\begin{pmatrix} h(\zeta) I_2 & 0 \\ 0 & h^{*}(\zeta) I_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \gamma_{+}(\zeta, 0) \gamma_{-}(\zeta, 0)^{-1} & 0 \\ 0 & \varphi_{+}(\zeta) \varphi_{-}(\zeta, 0)^{-1} \end{pmatrix}$$

are similar; hence they have the same eigenvalues. In particular,

$$(8.11) \quad h(\zeta) \in \left\{ \gamma_+(\zeta, 0)\gamma_-(\zeta, 0)^{-1}, \varphi_+(\zeta)\varphi_-(\zeta, 0)^{-1} \right\} \quad \text{if } \zeta \in U_+ \cap U_-.$$

Consider the open sets

$$V_\gamma := \{ \zeta \in U_+ \cap U_- \mid h(\zeta) = \gamma_+(\zeta, 0)\gamma_-(\zeta, 0)^{-1} \}$$

and

$$V_\varphi := \{ \zeta \in U_+ \cap U_- \mid h(\zeta) = \varphi_+(\zeta, 0)\varphi_-(\zeta, 0)^{-1} \}.$$

Then, by (8.11),  $V_\gamma \cup V_\varphi = U_+ \cap U_-$ . Hence, at least one of these sets, say  $V_\gamma$ , is non-empty. Since the functions  $h$  and  $\gamma_+(\cdot, 0)\gamma_-(\cdot, 0)^{-1}$  both are holomorphic on  $U_+ \cap U_-$  and  $U_+ \cap U_-$  is connected, it follows that  $h(\zeta) = \gamma_+(\zeta, 0)\gamma_-(\zeta, 0)^{-1}$  for all  $\zeta \in U_+ \cap U_-$ , which is not possible, by Lemma 8.1 (ii).  $\square$

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INSTITUT FÜR MATHEMATIK, HUMBOLDT-UNIVERSITÄT ZU BERLIN, RUDOWER CHAUSSEE 25,  
D-12489 BERLIN, GERMANY

*E-mail address:* leiterer@mathematik.hu-berlin.de